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DISPERSION OF NON-LINEAR SHALLOW WATER WAVES

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SUMMARY

Asymptotic solutions representing slowly varying wavetrains are obtained for two kinds of non-linear shallow water waves, i.e. Korteweg - de Vries waves (cnoidal waves) and Boussinesq waves.

Differential equations for slowly varying parameters are derived and it is shown that some of these equations can also be obtained by an averaging technique applied to the conservation equations of the problem. After an asymptotic expansion with respect to the small amplitude/depth ratio equations are given that determine the slow variations of amplitude, wavenumber, frequency, mean waveheight, etc. It is shown that these equations are hyperbolic.

A transformation of these equations into their characteristic form shows that two equations for wavenumber and energy density uncouple from the other ones. An explicit method of solution is indicated.

1. *Introduction.*

In the theory of wave propagation dispersion is an important phenomenon: an arbitrary initial disturbance of a wave system disperses into a slowly varying wavetrain after some time. This is illustrated very strikingly in the theory of linear waves which are governed by linear partial differential equations with constant coefficients. Because of the superposition principle for linear waves it is possible then to give the exact solution of the initial value problem as a Fourier integral and an asymptotic expansion for large time by means of the method of stationary phase gives a nearly uniform wave. Linear dispersive waves are discussed extensively by Eckart [1], Lighthill [2], Jeffreys [3], Peletier [4], Brillouin and Sommerfeld [5]. Dispersion is caused by the fact that in general for linear problems each uniform progressing wave is propagated with a velocity that depends on the wave-length and hence each component of a spectrum of waves propagates in a different way, causing the wavetrain to change its form continuously.

For linear wave equations with constant coefficients it is possible to solve the initial value problem exactly, but difficulties arise immediately when the coefficients are functions of the coordinates and time (for instance as a result of an inhomogeneous medium), or when the equations are non-linear. In these cases the superposition principle is not valid and in fact exact solutions are not available any more. Only asymptotic theories can give further information then.

The asymptotic theories that have been developed in recent years are concerned with finding solutions to wave problems representing slowly varying wavetrains, i.e. waves that are expected to have developed after a considerable time and that can be considered locally as nearly uniform. The asymptotic treatment is then based on the small variations of quantities like wavenumber, frequency, amplitude etc. within one wavelength or period. For linear problems, including the case of variable coefficients, J. B. Keller and his co-workers developed the so-called Ray-theory. In this connection we mention the papers of Lewis [6], Bleistein and Lewis [7] and Boersma [8].

For non-linear conservative problems recently several techniques were developed by Whitham [9, 10, 11, 12], which were refined for a special class of problems by Lighthill [13, 14]. In a first article Whitham [9] derives equations for slowly varying quantities such as wavenumber, frequency, amplitude etc. by considering a slowly varying wavetrain as locally uniform. Then for a uniform wavetrain the conserved quantities which occur in the conservation equations of the problem are averaged over one period. The averaged conserved quantities are now considered as functions of the coordinates and of time in the case of a slowly varying wavetrain and are substituted again into the conservation equations. This yields a set of equations for amplitude, frequency, wavenumber etc. as functions of the coordinates and of time.

In a second paper Whitham [10] introduces an averaged Lagrangian density method: for a uniform wave the Lagrangian density is averaged over one period. This averaged Lagrangian depends on parameters such as frequency, wavenumber, amplitude and also, depending on the order of the governing equations, on so-called pseudo-frequencies such as mean height, mean velocity, etc. In the case of a slowly varying wavetrain these parameters are considered again as functions of the coordinates and time and the Euler-Lagrange equations for the averaged Lagrangian variational principle yield a set of partial differential equations for the slowly varying parameters.

In this paper we use an asymptotic expansion for slowly varying wavetrains in order to obtain equations for slowly varying quantities for two kinds of non-linear shallow water waves, viz. Korteweg - de Vries ("cnoidal") waves and Boussinesq waves. A similar asymptotic series was used by Luke [15] in order to investigate asymptotic solutions of a non-linear Klein-Gordon equation. It is shown in this paper that the equations for the slowly varying parameters of the wavetrain can also be obtained by an appropriate averaging procedure applied to the conservation laws of the problem. This technique is essentially different from Whitham's averaging technique of conservation laws ([9]). The averaged equations are reduced further by an asymptotic expansion with respect to the amplitude/depth parameter. For both cnoidal and Boussinesq waves ultimately a hyperbolic set of two equations for the wavenumber and the energy is derived.

It is shown that these equations have the same form as the equations for the unsteady one-dimensional motion of a compressible gas with a fictitious pressure - density relation. A method of solution is given by transformation into an axisymmetric wave equation.

2. *Asymptotic representation of slowly varying wavetrains.*

The problem is to find asymptotic solutions of non-linear partial differential equations representing slowly varying wavetrains, i.e. waves that can be considered as nearly uniform in regions of order of magnitude of some wavelengths and periods. Taking the order of magnitude of the slow variations of wavelength, frequency, amplitude etc. as K^{-1}, with K large, we stretch the coordinates with this factor K in order to obtain a set of x, t - coordinates in which each unity of x and t contains a large number (of order K) of wavelengths and periods respectively.

For these coordinates lines of equal phase or wavefronts S(x, t) = constant can be defined as lines along which the normal derivative of the wave function u(x, t) is large (order K) compared to the tangential derivative (order unity). Accordingly it is assumed that the wave function u(x, t) of a slowly varying wavetrain can be represented asymptotically by

$$u(x,t) = U_1\left[KS(x,t),\ x,t\right] + \frac{1}{K}\,U_2\left[KS(x,t),\ x,t\right] + O\left(\frac{1}{K^2}\right).$$

Indeed the derivative of u normal to a line S(x, t) = constant is

$$\frac{\partial u}{\partial n} = \left|\nabla S\right|^{-1}.\ S_x\left[KS_x U_{1p} + U_{1x} + S_x U_{2p} + O\left(\frac{1}{K}\right)\right] +$$

$$+ \left|\nabla S\right|^{-1}.\ S_t\left[KS_t U_{1p} + U_{1t} + S_t U_{2p} + O\left(\frac{1}{K}\right)\right] = O(K),$$

where $p = KS(x,t)$, and the tangential derivative of u is

$$\frac{\partial u}{\partial s} = \left|\nabla S\right|^{-1}.\ (-S_t)\left[KS_x U_{1p} + U_{1x} + S_x U_{2p} + O\left(\frac{1}{K}\right)\right] +$$

$$+ \left|\nabla S\right|^{-1}.\ S_x\left[KS_t U_{1p} + U_{1t} + S_t U_{2p} + O\left(\frac{1}{K}\right)\right] = O(1).$$

It is also seen that in regions of order $1/K$ in the x, t - plane the variations of $p=KS(x,t)$ are of order unity. Hence the dependence of U_i on p describes the rapid local oscillations (it is anticipated that the dependence on p is oscillatory) and the dependence on x and t describes the large-scale variations of amplitude, frequency, wavelength, etc.

3. *The Korteweg - de Vries equation.*

Korteweg and de Vries [16] considered one-dimensional long waves in shallow water and derived an equation for the elevation $\bar{\eta}(\bar{x},\bar{t})$ of the water surface above the undisturbed depth h_o. This equation may be written as

$$\frac{\partial\bar{\eta}}{\partial\bar{t}} + \sqrt{gh_o}\left(1 + \frac{3\bar{\eta}}{2h_o}\right)\frac{\partial\bar{\eta}}{\partial\bar{x}} + \frac{1}{6}\sqrt{gh_o}.\, h_o^2\,\frac{\partial^3\bar{\eta}}{\partial\bar{x}^3} = 0. \qquad (3.1)$$

This equation is valid for small values of the two non-dimensional parameters ϵ and μ defined by

$$\epsilon = \frac{a}{h_o} \qquad \text{and} \qquad 6\mu = \frac{h_o^2}{\lambda_o^2},$$

where a is a typical amplitude and λ_o a typical wavelength.

Introducing non-dimensional variables x, t and η by means of

$$\bar{x} = \lambda_o x, \quad \frac{3\bar{\eta}}{2h_o} = \epsilon\eta, \quad \frac{\sqrt{gh_o}}{\lambda_o}\,\bar{t} = t,$$

eq. 3.1 transforms into

$$\eta_t + (1+\epsilon\eta)\eta_x + \mu\eta_{xxx} = 0. \qquad (3.2)$$

In eq. 3.2 the wave-height $\eta(x,t)$, the wave-length and period are all quantities of order of magnitude one.

It is worthwhile to give some attention to the linearized versions of eq. 3.2. If both ϵ and μ approach zero we get $\eta_t + \eta_x = 0$, which is a one-directional wave-equation with general solution

$$\eta(x,t) = F(x-t).$$

This means that an arbitrary disturbance of the free surface is propagated without distortion with non-dimensional velocity one in the positive x-direction.

This corresponds to the fully linearized shallow water theory in which no dispersion occurs. For our purposes the linearization with respect to ϵ only is more meaningful because the presence of the third order derivative yields a dispersive linear equation:

$$\eta_t + \eta_x + \mu\eta_{xxx} = 0. \tag{3.3}$$

Substitution of a uniform harmonic wave

$$\eta = A\cos\left[2\pi(\kappa x - \omega t)\right]$$

gives the dispersion relation for eq. 3.3:

$$\omega = \omega_0(\kappa) = \kappa - 4\pi^2\mu\kappa^3. \tag{3.4}$$

In order to investigate asymptotic solutions of eq. 3.2 representing a slowly varying wavetrain, the coordinates x and t are stretched with a large factor K:

$$x = K x^*, \qquad t = K t^*.$$

We then get after omission of the asterisks

$$\eta_t + (1 + \epsilon\eta)\eta_x + \frac{\mu}{K^2}\eta_{xxx} = 0. \tag{3.5}$$

Substitution of the asymptotic expansion for a slowly varying wavetrain:

$$\eta(x,t) = U\left[KS(x,t), x, t\right] + \frac{1}{K} V\left[KS(x,t), x, t\right] + O\left(\frac{1}{K^2}\right)$$

into eq. 3.5 yields (with $p=KS(x,t)$):

$$KS_t U_p + U_t + V_p S_t + (1 + \epsilon U + \frac{\epsilon}{K}V)(KS_x U_p + U_x + V_p S_x) +$$
$$+ \frac{\mu}{K^2}\left[K^3 S_x^3 U_{ppp} + K^2(3S_x^2 U_{ppx} + 3S_x S_{xx} U_{pp} + S_x^3 V_{ppp})\right] +$$
$$+ O(\tfrac{1}{K}) = 0. \tag{3.6}$$

Introducing the local wavenumber $\kappa = S_x$ and the local frequency $\omega = -S_t$ and equating to zero like powers of K in eq. 3.6, we get

$$O(K):\ (\kappa - \omega)U_p + \epsilon\kappa U U_p + \mu\kappa^3 U_{ppp} = 0, \tag{3.7}$$

$$O(1):\ (\kappa - \omega)V_p + \epsilon\kappa(UV_p + U_p V) + \mu\kappa^3 V_{ppp} = F(p,x,t), \tag{3.8}$$

with

$$F(p,x,t) = -U_t - U_x - \epsilon U U_x - 3\mu(\kappa^2 U_{ppx} + \kappa\kappa_x U_{pp}).$$

Equation 3.7 may be considered as an ordinary differential equation for U as a function of p with coefficients depending on x and t. Integrating eq. 3.7 twice with respect to p we find successively:

$$(\kappa - \omega)U + \tfrac{1}{2}\epsilon\kappa U^2 + \mu\kappa^3 U_{pp} = \tfrac{1}{2}\alpha,$$
$$\mu\kappa^3 U_p^2 = \alpha U + \beta + (\omega - \kappa)U^2 - \tfrac{1}{3}\epsilon\kappa U^3, \tag{3.9}$$

with α and β unknown functions of x and t figuring as constants of integration. From eq. 3.9 can be seen that U is a periodic function of p oscillating between two zeroes U_1 and U_2 of the righthand side of eq. 3.9. In general the righthand side of eq. 3.9 has three zeroes. U_1 and U_2 are determined by the requirement that the righthand side of eq. 3.9 has to be positive for $U_1 < U < U_2$. The dependence of U on p can be given in implicit form as

$$p + \gamma = (\mu\kappa^3)^{\frac{1}{2}} \int_{U_1}^{U} \left[\alpha U + \beta + (\omega - \kappa)U^2 - \frac{1}{3}\epsilon\kappa U^3\right]^{-\frac{1}{2}} dU,$$

with $\gamma(x,t)$ as a non-essential shifting constant. The period of U must be independent of x and t, otherwise differentiation of U with respect to x or t would result in unbounded terms for large p. Let us normalize the period of U to unity:

$$\frac{1}{2} = (\mu\kappa^3)^{\frac{1}{2}} \int_{U_1}^{U_2} \left[\alpha U + \beta + (\omega - \kappa)U^2 - \frac{1}{3}\epsilon\kappa U^3\right]^{-\frac{1}{2}} dU. \tag{3.10}$$

For a uniform wavetrain ω, κ, α and β are independent of x and t and eq. 3.10 reduces to an algebraical dispersion relation. For $\epsilon=0$ we have $\alpha=0$ and $\beta=\kappa-\omega$ and then eq. 3.10 becomes

$$\frac{1}{2} = \sqrt{\frac{\mu\kappa^3}{\kappa-\omega}} \int_{-1}^{+1} \frac{dU}{\sqrt{1-U^2}} = \pi\sqrt{\frac{\mu\kappa^3}{\kappa-\omega}},$$

which is equivalent to the dispersion relation (eq. 3.4) for the linearized problem.

We see that there are four slowly varying functions in the problem: $\omega(x,t)$, $\kappa(x,t)$, $\alpha(x,t)$ and $\beta(x,t)$, and we are interested in finding four equations to determine them. Equation 3.10 together with the relation

$$\omega_x + \kappa_t = 0 \tag{3.11}$$

provide two of them. The two remaining equations are obtained from the condition of boundedness of the second term $V(p,x,t)$ in the asymptotic series for $\eta(x,t)$.

Let us write eq. 3.8 as

$$\frac{\partial}{\partial p}\left[(\kappa-\omega)V + \epsilon\kappa UV + \mu\kappa^3 V_{pp}\right] = F(p,x,t). \tag{3.12}$$

The term between square brackets in eq. 3.12 has to be bounded so

$$\int^{p} F dp' + p_1$$

must be bounded, with p_1 a constant of integration. Because $F(p,x,t)$ is periodic in p with period 1, this integral is bounded for large p only if

$$\int_0^1 F dp = 0, \tag{3.13}$$

which may be simplified as

$$\int_0^1 \left[U_t + U_x + \epsilon UU_x + 3\mu(\kappa^2 U_{ppx} + \kappa\kappa_x U_{pp})\right] dp = \int_0^1 \left[U_t + U_x + \epsilon UU_x\right] dp = 0, \tag{3.14}$$

by making use of the periodicity of U_p and U_{px}.

Equation 3.14 is the third equation containing ω, κ, α and β. The fourth equation follows after integration of eq. 3.12 and noting that $V = U_p$ is a solution of the corresponding homogenous equation. Putting $V = wU_p$, we can write

$$\mu\kappa^3 \frac{\partial}{\partial p}\left(U_p^2 w_p\right) = \mu\kappa^3 \frac{\partial}{\partial p}\left[V_p U_p - V U_{pp}\right] = U_p\left\{\int^p F dp' + p_1\right\}.$$

With the same reasoning as above we have that the term between square brackets must be bounded and because of the periodicity of the inhomogeneous term, we have

$$\int_0^1 U_p \left[\int^p F dp' + p_1\right] dp = 0.$$

By partial integration we get

$$\left[U\int^p F dp' + Up_1\right]_0^1 - \int_0^1 UF dp = -\int_0^1 UF dp = 0,$$

or

$$\int_0^1 U\left[U_t + U_x + \epsilon U U_x + 3\mu\left(\kappa^2 U_{ppx} + \kappa\kappa_x U_{pp}\right)\right] dp = 0. \qquad (3.15)$$

The four equations 3.10, 3.11, 3.14 and 3.15 determine the four functions $\omega(x,t)$, $\kappa(x,t)$, $\alpha(x,t)$ and $\beta(x,t)$. In the next section we will show that eqs. 3.14 and 3.15 also can be obtained by an appropriate averaging of the conservation laws of the problem. In section 5 we will give an asymptotic approximation respect to ϵ that enables us to simplify the equations considerably.

4. *Conservation laws for cnoidal waves.*

Several conservation laws can be derived from the governing equation 3.5. Equation 3.5 itself is a conservative equation:

$$\frac{\partial}{\partial t}(\eta) + \frac{\partial}{\partial x}\left(\eta + \tfrac{1}{2}\epsilon\eta^2 + \frac{\mu}{K^2}\eta_{xx}\right) = 0. \qquad (4.1)$$

A second conservation law is constructed by multiplication with η:

$$\eta\frac{\partial}{\partial t}(\eta) + \eta\frac{\partial}{\partial x}\left(\eta + \tfrac{1}{2}\epsilon\eta^2 + \frac{\mu}{K^2}\eta_{xx}\right) = 0, \qquad (4.2)$$

or in conservative form:

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\eta^2\right) + \frac{\partial}{\partial x}\left[\tfrac{1}{2}\eta^2 + \frac{1}{3}\epsilon\eta^3 + \frac{\mu}{K^2}\left(\eta\eta_{xx} - \tfrac{1}{2}\eta_x^2\right)\right] = 0. \qquad (4.3)$$

Higher order conservation laws are obtained by multiplication of η^2, η^3, etc., but they will not be needed here. Equation 4.1 may be considered as an approximate form of the equation of conservation of mass and eq. 4.3 as the corresponding approximation to the equation of conservation of energy.

Consider now a uniform wavetrain:

$$\eta(x,t) = U\left[K(\kappa x - \omega t);\, \kappa, \omega, \alpha, \beta\right], \qquad (4.4)$$

satisfying eq. 3.7 with $p = K(\kappa x - \omega t)$:

$$(\kappa-\omega)\,U_p + \epsilon\kappa\,U\,U_p + \mu\kappa^3\,U_{ppp} = 0, \tag{4.5}$$

with constants $\kappa,\omega,\alpha,\beta$ satisfying the dispersion relation 3.10. It is assumed that a slowly varying wavetrain can be represented locally, i.e. in regions of order K^{-1} in x, t-plane, by the uniform wavetrain solution 4.4 of eq. 4.5 but with slowly varying parameters $\omega(x,t)$, $\kappa(x,t)$, $\alpha(x,t)$ and $\beta(x,t)$, that still satisfy the dispersion relation 3.10. So we put:

$$\eta(x,t) = U\left[p;\ \alpha(x,t),\ \beta(x,t),\ \kappa(x,t),\ \omega(x,t)\right]. \tag{4.6}$$

with $p_x = K\kappa(x,t)$ and $p_t = -K\omega(x,t)$. Introducing the notation

$$U_x = U_\alpha \alpha_x + U_\beta\,\beta_x + U_\kappa\,\kappa_x + U_\omega\,\omega_x,$$

and similarly for U_t, we get after substitution of eq. 4.6 into the conservation equations 4.1 and 4.2:

$$- K\omega U_p + U_t + (1+\epsilon U)(K\kappa\,U_p + U_x) +$$

$$+ \frac{\mu}{K^2}\left[K^3\kappa^3\,U_{ppp} + K^2(3\kappa^2 U_{ppx} + 3\,\kappa\,\kappa_x U_{pp})\right] + O\left(\frac{1}{K}\right) = 0, \tag{4.7}$$

and

$$U\left[-K\omega U_p + U_t + (1+\epsilon U)(K\kappa\,U_p + U_x) +\right.$$

$$\left.+ \frac{\mu}{K^2}\left\{K^3\kappa^3 U_{ppp} + K^2(3\kappa^2 U_{ppx} + 3\,\kappa\,\kappa_x U_{pp})\right\}\right] + O\left(\frac{1}{K}\right) = 0. \tag{4.8}$$

By virtue of eq. 4.5 the terms of order K in eqs. 4.7 and 4.8 vanish, and the remaining parts are averaged over one period in p in order to get equations for the x, t - dependence of the wavetrain:

$$\int_0^1 \left[U_t + U_x + \epsilon\,U\,U_x + 3\mu(\kappa^2 U_{ppx} + \kappa\,\kappa_x U_{pp})\right] dp + O\left(\frac{1}{K}\right) =$$

$$= \int_0^1 F dp + O\left(\frac{1}{K}\right) = 0,$$

$$\int_0^1 U\left[U_t + U_x + \epsilon U U_x + 3\mu(\kappa^2 U_{ppx} + \kappa\,\kappa_x\,U_{pp})\right] dp + O\left(\frac{1}{K}\right) =$$

$$= \int_0^1 UF dp + O\left(\frac{1}{K}\right) = 0.$$

These two relations are, apart from terms $O(1/K)$ that can be omitted within this order of approximation, in exact agreement with the relations 3.14 and 3.15 of the last section.

5. *Asymptotic expansion with respect to* ϵ.

In this section we will derive equations for wavenumber, amplitude, frequency and mean height that follow after asymptotic expansion with respect to ϵ, viz. in the case of waves of small finite amplitude.

At first we expand $U[p,x,t]$ in a Fourier series in p with coefficients depending on x and t. To this end we introduce new functions $\eta_0(x,t)$, $A(x,t)$ and $\Phi[p,x,t]$ by means of

$$U[p,x,t] = \eta_0(x,t) + A(x,t)\,\Phi[p,x,t],$$
$$\eta_0(x,t) = \tfrac{1}{2}\left[U_{max}(x,t) + U_{min}(x,t)\right]. \qquad (5.1)$$

Now $A(x,t)$ has the significance of an amplitude function and $\Phi[p,x,t]$ is a periodic function in p with period 1. The introduction of $\eta_0(x,t)$ allows us to fix the extremal values of Φ as -1 and +1. Note that η_0 is not equal to the mean waveheight, i.e the averaged value of U over one period. For computational reasons the function $\eta_0(x,t)$ has advantages and only in a later stage we will switch to the mean waveheight.

It is possible now to eliminate the functions $\alpha(x,t)$ and $\beta(x,t)$ that have no physical meaning and have them replaced by $\eta_0(x,t)$ and $A(x,t)$. Inserting eq. 5.1 into eq. 3.9, we obtain

$$\mu\kappa^3 A^2\Phi_p^2 = \alpha(\eta_0+A\Phi) + \beta + (\omega-\kappa)(\eta_0+A\Phi)^2 - \frac{1}{3}\epsilon\kappa(\eta_0+A\Phi)^3. \qquad (5.2)$$

Using the fact that $\Phi_p = 0$ for both $\Phi = -1$ and $\Phi = +1$, we arrive at two algebraical equations from which α and β can be solved:

$$\alpha(\eta_0+A) + \beta + (\omega-\kappa)(\eta_0+A)^2 - \frac{1}{3}\epsilon\kappa(\eta_0+A)^3 = 0,$$
$$\alpha(\eta_0-A) + \beta + (\omega-\kappa)(\eta_0-A)^2 - \frac{1}{3}\epsilon\kappa(\eta_0-A)^3 = 0.$$

Solving α and β and substitution into eq. 5.2 yields the differential equation for $\Phi(p,x,t)$:

$$\mu\kappa^3\Phi_p^2 = (1-\Phi^2)\left(\frac{1}{3}\epsilon A\kappa\Phi + \epsilon\kappa\eta_0 + \kappa-\omega\right). \qquad (5.3)$$

Before proceeding we fix the order of magnitude of $A(x,t)$ and $\eta_0(x,t)$. It is clear that A is of order unity. For $\epsilon \to 0$ we get the linear problem for which $\eta_0(x,t)$ is trivial: the disturbances in the linear case can be taken symmetrical with respect to the undisturbed state. For $\epsilon \neq 0$ it is assumed that $\eta_0(x,t)$ is of order ϵ and hence we put $\eta_0(x,t) = \epsilon\eta_1(x,t)$, with $\eta_1(x,t) = 0(1)$ Equation 5.3 becomes

$$\mu\kappa^3\Phi_p^2 = (1-\Phi^2)\left(\frac{1}{3}\epsilon\kappa A\Phi + \epsilon^2\kappa\eta_1 + \kappa-\omega\right). \qquad (5.4)$$

The condition that Φ has period 1 in p yields the dispersion relation

$$\tfrac{1}{2} = (\mu\kappa^3)^{\frac{1}{2}}\int_{-1}^{+1}\left[(1-\Phi^2)\left(\frac{1}{3}\epsilon A\kappa\Phi + \epsilon^2\kappa\eta_1 + \kappa-\omega\right)\right]^{-\frac{1}{2}} d\Phi. \qquad (5.5)$$

We now apply Lindstedt's method (Minorsky [17]) in order to obtain an asymptotic expansion of solutions of eq. 5.4 in the form of a Fourier series in p. Differentiation of eq. 5.4 with respect to p and introduction of

$$q = p\sqrt{\frac{\kappa-\omega}{\mu\kappa^3}}$$

yields the equation:

$$(\kappa-\omega)(\Phi_{qq}+\Phi) = -\tfrac{1}{2}\epsilon A\kappa\Phi^2 - \epsilon^2\eta_1\kappa\Phi + \frac{1}{6}\epsilon A\kappa. \qquad (5.6)$$

For $\epsilon \to 0$ we have the linearized solution $\Phi = \cos q$, because, according to eq. 5.4, we only can take one elementary solution of $\Phi_{qq} + \Phi = 0$. Essential for Lindstedt's method is that both Φ and q are considered as functions of a new variable s and both are expanded in a power series in ϵ:

$$\Phi(s) = \cos s + \epsilon\,\Phi_1(s) + \epsilon^2\,\Phi_2(s) + \ldots, \tag{5.7}$$

$$q = s(1 + c_1\epsilon + c_2\epsilon^2 + \ldots). \tag{5.8}$$

The introduction of the unknown constants c_1, c_2, allows us to avoid secular terms in the series expansion for $\Phi(s)$, viz. terms that are not periodic in s. We have the boundary conditions

$$\Phi_1(0) = \Phi_2(0) = \Phi_3(0) = \ldots. = 0,$$

$$\Phi_1'(0) = \Phi_2'(0) = \Phi_3'(0) = \ldots = 0.$$

Substitution of eqs. 5.7 and 5.8 into eq. 5.6 yields

$$-\cos s + \epsilon\Phi_1'' + \epsilon^2\Phi_2'' + \ldots. + \left[1 + 2\epsilon c_1 + \epsilon^2(c_1^2 + 2c_2) + \ldots.\right] \times$$

$$\times\left[\cos s + \epsilon\Phi_1 + \epsilon^2\Phi_2 + \ldots\right] = \left[1 + 2\epsilon c_1 + \epsilon^2(c_1^2+2c_2) + \ldots\right]\left[\frac{A\kappa\epsilon}{6(\kappa-\omega)} - \frac{\epsilon^2\eta_1\kappa}{\kappa-\omega}\times\right.$$

$$\left.\times(\cos s + \epsilon\Phi_1 + \epsilon^2\Phi_2 + \ldots) - \frac{\epsilon A\kappa}{2(\kappa-\omega)}(\cos^2 s + 2\epsilon\Phi_1\cos s + \ldots)\right]. \tag{5.9}$$

The lowest order term vanishes and the coefficient of ϵ yields a differential equation for $\Phi_1(s)$:

$$\Phi_1''(s) + \Phi_1(s) = -2c_1\cos s + \frac{A\kappa}{6(\kappa-\omega)} - \frac{A\kappa}{2(\kappa-\omega)}\cos^2 s. \tag{5.10}$$

The term $-2c_1\cos s$ in the righthand side would give rise to a term proportional to s cos s in $\Phi_1(s)$. This would be a secular term destroying the periodicity of $\Phi_1(S)$. Hence we take $c_1=0$. The solution of eq. 5.10 satisfying the boundary conditions is:

$$\Phi_1(s) = -\frac{A\kappa}{12(\kappa-\omega)}(1-\cos 2s).$$

The differential equation for $\Phi_2(s)$ is:

$$\Phi_2''(s) + \Phi_2(s) = \left[-\frac{\eta_1\kappa}{\kappa-\omega} - 2c_2 + \frac{A^2\kappa^2}{24(\kappa-\omega)^2}\right]\cos s - \frac{A^2\kappa^2}{24(\kappa-\omega)^2}\cos 3s. \tag{5.11}$$

Here again the term with cos s is a secular term and hence its coefficient must vanish:

$$c_2 = \frac{A^2\kappa^2}{48(\kappa-\omega)^2} - \frac{\eta_1\kappa}{2(\kappa-\omega)}.$$

After solving $\Phi_2(s)$ and proceeding to the equation for $\Phi_3(s)$, yielding $c_3=0$, we get the following asymptotic solution of eq. 5.6:

$$\Phi(s) = \cos s - \frac{\epsilon A\kappa}{12(\kappa-\omega)}(1-\cos 2s) + \frac{\epsilon^2 A^2\kappa^2}{192(\kappa-\omega)^2}(\cos 3s - \cos s) +$$

$$+ O(\epsilon^3),$$

with:

$$q = p\sqrt{\frac{\kappa-\omega}{\mu\kappa^3}} = s\left[1 + \epsilon^2\left\{\frac{A^2\kappa^2}{48(\kappa-\omega)^2} - \frac{\eta_1\kappa}{2(\kappa-\omega)}\right\} + O(\epsilon^4)\right]. \tag{5.12}$$

The condition that $\Phi(p)$ is periodic with period 1 gives $s = 2\pi p$ and from eq. 5.12 then follows the dispersion relation:

$$\kappa-\omega = 4\pi^2\mu\kappa^3\left[1 + \epsilon^2\left\{\frac{A^2\kappa^2}{24(\kappa-\omega)^2} - \frac{\eta_1\kappa}{\kappa-\omega}\right\}\right] + O(\epsilon^4). \tag{5.13}$$

This relation also could have been obtained by expansion of the complete dispersion relation (eq. 5.5) with respect to ϵ. Eq. 5.13 is written more conveniently as:

$$\omega = \kappa - 4\pi^2\mu\kappa^3 + \epsilon^2\left[\eta_1\kappa - \frac{A^2}{96\pi^2\mu\kappa}\right] + O(\epsilon^4), \tag{5.14}$$

and similarly we have for $U(p,x,t)$:

$$U(p,x,t) = \epsilon\eta_1 + A\cos(2\pi p) + \frac{\epsilon A^2}{48\pi^2\mu\kappa^2}\left[\cos(4\pi p) - 1\right] + O(\epsilon^2).$$

Introducing the mean value $\epsilon h_1(x,t)$ of $U(p,x,t)$ by

$$\epsilon h_1(x,t) = \epsilon\eta_1 - \frac{\epsilon A^2}{48\pi^2\mu\kappa^2} + O(\epsilon^2), \quad h_1 = O(1),$$

we have

$$U(p,x,t) = \epsilon h_1 + A\cos(2\pi p) + \frac{\epsilon A^2}{48\pi^2\mu\kappa^2}\cos(4\pi p) + O(\epsilon^2), \tag{5.15}$$

$$\omega = \omega_0(\kappa) + \epsilon^2\left[\kappa h_1 + \frac{A^2}{96\pi^2\mu\kappa}\right] + O(\epsilon^4), \tag{5.16}$$

with the abbreviation:

$$\omega_0(\kappa) = \kappa - 4\pi^2\mu\kappa^3.$$

This is the ultimate form of $U(p,x,t)$ that can be used for substitution into the integral relations 3.14 and 3.15.
Writing eq. 3.14 as

$$\frac{\partial}{\partial t}\left[\int_0^1 U\,dp\right] + \frac{\partial}{\partial x}\left[\int_0^1 U\,dp\right] + \frac{\partial}{\partial x}\left[\tfrac{1}{2}\epsilon\int_0^1 U^2\,dp\right] = 0,$$

and eq. 3.15 as

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\int_0^1 U^2\,dp\right] + \frac{\partial}{\partial x}\left[\tfrac{1}{2}\int_0^1 U^2\,dp\right] + \frac{\partial}{\partial x}\left[\tfrac{1}{3}\epsilon\int_0^1 U^3\,dp\right] +$$
$$+ 3\mu\kappa^2\left[\int_0^1 U\,U_{ppx}\,dp\right] + 3\mu\kappa\kappa_x\left[\int_0^1 U\,U_{pp}\,dp\right] = 0,$$

we get, respectively, after substitution of eq. 5.15:

$$\frac{\partial h_1}{\partial t} + \frac{\partial h_1}{\partial x} + \frac{\partial}{\partial x}(\tfrac{1}{4}A^2) + O(\epsilon^2) = 0,$$

$$\frac{\partial}{\partial t}(A^2) + \frac{\partial}{\partial x}\left[(1-12\pi^2\mu\kappa^2)A^2\right] + O(\epsilon^2) = 0.$$

Introducing $H_o = \epsilon^2 h_1$ and $E = \epsilon^2 A^2$, these last two equations, together with the dispersion relation can be recasted as follows:

$$\omega = \omega_0(\kappa) + \kappa H_o + \frac{E}{96\pi^2\mu\kappa}, \quad \omega_o(\kappa) = \kappa - 4\pi^2\mu\kappa^3, \tag{5.17}$$

$$\frac{\partial H_o}{\partial t} + \frac{\partial}{\partial x}\left[H_o + \tfrac{1}{4}E\right] = 0, \tag{5.18}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\omega_o'(\kappa)E\right] = 0. \tag{5.19}$$

Equation 5.18 may be considered as an averaged equation of conservation of mass and eq. 5.19 expresses the conservation of the "averaged energy" E of the wavetrain, which is propagated approximately with the linear group velocity $C_o = \omega_o'(\kappa)$. In the next section these equations will be investigated further.

6. *The equations for κ, ω, E and H_o.*

In order to find the characteristic velocities of the set of equations 5.17, 5.18, 5.19 we transform equation 5.17 by differentiation with respect to x and after using 3.11 we then get:

$$\kappa_t + \omega_o'(\kappa)\kappa_x + \kappa_x H_o + \kappa H_{ox} + \frac{E_x}{96\pi^2\mu\kappa} - \frac{E\kappa_x}{96\pi^2\mu\kappa^2} = 0. \tag{6.1}$$

Multiplying eqs. 5.18 and 5.19 by constants λ and ν respectively and adding them to eq. 6.1, the condition that κ, H_o and E are to be differentiated in the same characteristic direction C gives a set of algebraical equations for C, λ and ν:

$$C = \omega_o'(\kappa) + H_o - \frac{E}{96\pi^2\mu\kappa^2} + \nu\omega_o''(\kappa)E,$$

$$\lambda C = \kappa + \lambda,$$

$$\nu C = \frac{1}{96\pi^2\mu\kappa} + \tfrac{1}{4}\lambda + \nu\omega_o'(\kappa).$$

Elimination of λ and ν gives one equation for C:

$$\left[C-\omega_o'(\kappa)\right]^2 = \left[H_o - \frac{E}{96\pi^2\mu\kappa^2}\right]\left[C-\omega_o'(\kappa)\right] + \omega_o''(\kappa)E\left[\frac{\kappa}{4(C-1)} + \frac{1}{96\pi^2\mu\kappa}\right]. \tag{6.2}$$

It is seen from eq. 6.2 that for linear waves with $H_o \to 0$ and $E \to 0$ there is a double root $C = C_o = \omega_o'(\kappa)$, viz. the linear group velocity. For non-linear waves we have $H_o = O(\epsilon^2)$ and $E = O(\epsilon^2)$ and then two roots C_1 and C_2 lie near $C_0 = \omega_0'(\kappa)$ and one root C_3 near $C=1$. The three characteristic velocities C_i $(i=1,2,3)$ are approximately found to be:

$$C_{1,2} = C_0 \pm \sqrt{\omega_0''(\kappa) E \left[\frac{\kappa}{4(C_0-1)} + \frac{1}{96\pi^2\mu\kappa}\right]} + O(\epsilon^{\frac{3}{2}}) =$$

$$= C_0 \pm \tfrac{1}{2}\sqrt{E} + O(\epsilon^{\frac{3}{2}}), \tag{6.3}$$

$$C_3 = 1 + \frac{\kappa E \omega_0''(\kappa)}{4(1-C_0)^2} + O(\epsilon^3) =$$

$$= 1 - \frac{E}{24\pi^2\mu\kappa^2} + O(\epsilon^3). \tag{6.4}$$

The characteristic velocities are real and hence we are dealing with a hyperbolic system of equations, that can be written as follows:

$$\left[\frac{\partial\kappa}{\partial t} + C_i\frac{\partial\kappa}{\partial x}\right] + \lambda_i\left[\frac{\partial H_0}{\partial t} + C_i\frac{\partial H_0}{\partial x}\right] + \nu_i\left[\frac{\partial E}{\partial t} + C_i\frac{\partial E}{\partial x}\right] = 0, \quad i=1,2,3. \tag{6.5}$$

with multipliers λ_i and ν_i given by

$$\lambda_i = \frac{\kappa}{C_i-1},$$

$$\nu_i = \frac{1}{C_i-C_0}\left[\frac{1}{96\pi^2\mu\kappa} + \frac{\kappa}{4(C_i-1)}\right]. \tag{6.6}$$

Note that the characteristic velocities C_i do not depend on the mean elevation H_0 to this order of approximation. Also we note that for i=1 and i=2 (corresponding to the characteristic velocities C_1 and C_2 which lie near the linear group velocity $C_0 = \omega_0'(\kappa)$) the multipliers ν_1 and ν_2 are large of order ϵ^{-1} compared to λ_1 and λ_2 because of $C_{1,2} = C_0 \pm O(\epsilon)$. Because of the fact that H_0 and E both are of order ϵ^2, the terms containing H_0 in eq. 6.5. can be omitted for i=1 and i=2. In fact they are of the same order of magnitude as terms omitted in earlier etages of the analysis. Hence to the present order of approximation we get *two* equations for the *two* functions κ and E:

$$\frac{\partial\kappa}{\partial t} + (C_0+\tfrac{1}{2}\sqrt{E})\frac{\partial\kappa}{\partial x} - \frac{1}{48\pi^2\mu\kappa\sqrt{E}}\left[\frac{\partial E}{\partial t} + (C_0+\tfrac{1}{2}\sqrt{E})\frac{\partial E}{\partial x}\right] = 0,$$

$$\frac{\partial\kappa}{\partial t} + (C_0-\tfrac{1}{2}\sqrt{E})\frac{\partial\kappa}{\partial x} + \frac{1}{48\pi^2\mu\kappa\sqrt{E}}\left[\frac{\partial E}{\partial t} + (C_0-\tfrac{1}{2}\sqrt{E})\frac{\partial E}{\partial x}\right] = 0.$$

By adding and subtracting these equations we find successively

$$\frac{\partial\kappa}{\partial t} + C_0\frac{\partial\kappa}{\partial x} - \frac{1}{96\pi^2\mu\kappa}\frac{\partial E}{\partial x} = 0, \tag{6.7}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(C_0 E) = 0. \tag{6.8}$$

Making use of $C_0(\kappa) = 1-12\pi^2\mu\kappa^2$, we introduce C_0 as a new dependent variable:

$$\frac{\partial C_o}{\partial t} + C_o \frac{\partial C_o}{\partial x} + \frac{1}{4} \frac{\partial E}{\partial x} = 0, \tag{6.9}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(C_o E) = 0. \tag{6.10}$$

This remarkable result shows that for cnoidal waves the "energy" $E(x,t)$ and its propagation velocity $C_o(x,t)$ satisfy the one-dimensional unsteady equations for a compressible gas:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0,$$

with an "adiabatic law" $p = \rho^2/8$. These equations are also similar to the first order non-linear shallow water equations.

7. *The Boussinesq equations*

The Boussinesq equations for one-dimensional shallow water waves are

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + g\frac{\partial \bar{h}}{\partial \bar{x}} + \frac{1}{3}\bar{h}_o \frac{\partial^3 \bar{h}}{\partial \bar{x}\,\partial \bar{t}^2} = 0, \tag{7.1}$$

$$\frac{\partial \bar{h}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{h}}{\partial \bar{x}} + \bar{h}\frac{\partial \bar{u}}{\partial \bar{x}} = 0, \tag{7.2}$$

with $\bar{u}(\bar{x},\bar{t})$ the horizontal velocity which is taken independent from the depth, $\bar{h}(\bar{x},\bar{t})$ the total depth and $\bar{h}_o$ the undisturbed depth. An interesting derivation of eqs. 7.1 and 7.2 is given by Whitham [12].

Just as for the Korteweg-de Vries equation these equations hold for small values of the two parameters ϵ and μ defined by

$$\epsilon = \frac{a}{\bar{h}_o}, \quad \mu = \frac{\bar{h}_o^2}{3\lambda_o^2}.$$

Introducing non-dimensional variables h, u, x and t:

$$\bar{x} = \lambda_o x, \quad \frac{\sqrt{g\bar{h}_o}}{\lambda_o}\bar{t} = t, \quad \bar{h} = \bar{h}_o h, \quad \bar{u} = u\sqrt{g\bar{h}_o},$$

we arrive at

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} + \mu\frac{\partial^3 h}{\partial x\,\partial t^2} = 0, \tag{7.3}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0. \tag{7.4}$$

Now in eqs. 7.3 and 7.4 we have wavelengths and periods of order unity, $h = 1 + O(\epsilon)$ and $u = O(\epsilon)$ Linearization with respect to ϵ and μ yields the fully linearized shallow water equations that display no dispersion:

$$u_t + h_x = 0; \; h_t + u_x = 0.$$

Linearization with respect to ϵ only gives the dispersive linear system:

$$u_t + h_x + \mu h_{xtt} = 0,$$

$$h_t + u_x = 0.$$

Substitution of the harmonic wave

$$u = B \cos \left[2\pi(\kappa x - \omega t)\right],$$

$$h = 1 + A \cos \left[2\pi(\kappa x - \omega t)\right],$$

gives the dispersion relations for the linear waves

$$\kappa^2 - \omega^2 = 4\pi^2 \mu \kappa^2 \omega^2,$$

$$A\omega - B\kappa = 0.$$

Stretching the coordinates x and t with a large factor K by means of $x = Kx^*$ and $t = Kt^*$, we get after omission of the asterisks:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} + \frac{\mu}{K^2} \frac{\partial^3 h}{\partial x \partial t^2} = 0, \tag{7.5}$$

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0. \tag{7.6}$$

In order to substitute asymptotic series for $u(x,t)$ and $h(x,t)$ we note that it is natural to expect that u and h have the same curves $S(x,t)$ = constant as wavefronts and accordingly we put:

$$u(x,t) = U\left[KS(x,t), x, t\right] + \frac{1}{K} U_1\left[KS(x,t), x, t\right] + O\left(\frac{1}{K^2}\right),$$

$$h(x,t) = H\left[KS(x,t), x, t\right] + \frac{1}{K} H_1\left[KS(x,t), x, t)\right] + O\left(\frac{1}{K^2}\right).$$

Substitution of these two asymptotic expansions into eqs. 7.5 and 7.6 yields as terms of order K:

$$-\omega U_p + \kappa U U_p + \kappa H_p + \mu\kappa\omega^2 H_{ppp} = 0, \tag{7.7}$$

$$-\omega H_p + \kappa U H_p + \kappa H U_p = 0, \tag{7.8}$$

with $\kappa = S_x$, $\omega = -S_t$ and $p = KS(x,t)$.

The coefficients of order unity yield

$$(\kappa U - \omega)U_{1p} + \kappa U_p U_1 + \kappa H_{1p} + \mu\kappa\omega^2 H_{1ppp} = F_1(p,x,t), \tag{7.9}$$

with

$$F_1(p,x,t) = -U_t - U U_x - H_x - 2\omega\omega_x \mu H_{pp} +$$
$$+ \mu\kappa\omega_t H_{pp} + 2\kappa\omega\mu H_{ppt} - \omega^2 \mu H_{ppx},$$

and

$$(\kappa U - \omega)H_{1p} + \kappa U_p H_1 + \kappa U_1 H_p + \kappa H U_{1p} = F_2(p,x,t), \tag{7.10}$$

with

$$F_2(p,x,t) = -H_t - UH_x - HU_x.$$

Let us give attention to eqs. 7.7 and 7.8 first. Integration with respect to p gives

$$-\omega U + \tfrac{1}{2}\kappa U^2 + \kappa H + \mu\kappa\omega^2 H_{pp} = \alpha - \frac{\omega^2}{2\kappa}, \qquad (7.11)$$

$$-\omega H + \kappa H U = \beta. \qquad (7.12)$$

Elimination of U from eq. 7.11 by means of eq. 7.12 and then integration with respect to p results in an equation for H only

$$\mu\kappa\omega^2 H_p^2 = \alpha H + \gamma + \frac{\beta^2}{\kappa H} - \kappa H^2, \qquad (7.13)$$

with $\alpha(x,t)$, $\beta(x,t)$ and $\gamma(x,t)$ as constants of integration. From eq. 7.13 follows that H is a periodic function of p oscillating between two zeroes of the righthand side of eq. 7.13. By virtue of eq. 7.12 we have that U also is periodic in p with the same period as H and from eq. 7.8 it is deduced that H_p and U_p are zero simultaneously and hence H and U are oscillating in phase. The condition that H and U have a period that is independent of x and t and that can be normalized to unity, gives the dispersion relation

$$\tfrac{1}{2} = (\mu\kappa\omega^2)^{\frac{1}{2}} \int_{H_{min}}^{H_{max}} \left[\alpha H + \gamma + \frac{\beta^2}{\kappa H} - \kappa H^2\right]^{-\frac{1}{2}} dH. \qquad (7.14)$$

The equation $\omega_x + \kappa_t = 0$ together with eq. 7.14 provide two equations for the set of five needed for the five unknown slowly varying functions $\omega(x,t)$, $\kappa(x,t)$, $\alpha(x,t)$, $\beta(x,t)$ and $\gamma(x,t)$ The three remaining equations are obtained again by imposing conditions of boundedness on the higher order terms U_1 and H_1 in the asymptotic expansions of U and H.

Eqs. 7.9 and 7.10 which determine U_1 and H_1 can be integrated once with respect to p:

$$(\kappa U - \omega)U_1 + \kappa H_1 + \mu\kappa\omega^2 H_{1pp} = \int^p F_1\, dp' + \gamma_1 = G_1(p,x,t), \qquad (7.15)$$

$$(\kappa U - \omega)H_1 + \kappa H U_1 = \int^p F_2\, dp' + \gamma_2 = G_2(p,x,t), \qquad (7.16)$$

with γ_1 and γ_2 as constants of integration. By the same reasoning as in sec. 3 we have that G_1 and G_2 must be bounded for large p and because of the periodicity of F_1 and F_2 we have:

$$\int_0^1 F_1\, dp = 0, \qquad (7.17)$$

and

$$\int_0^1 F_2\, dp = 0. \qquad (7.18)$$

A third integral relation follows after integration of eqs. 7.15 and 7.16. Noting that $U_1 = U_p$ and $H_1 = H_p$ are solutions of the homogeneous equations corresponding to eqs. 7.15 and 7.16, we have after putting $H_1 = wH_p$ and elimination of U_1:

$$-(\kappa U - \omega)^2 w H_p + \kappa^2 w H H_p + \mu\kappa^2\omega^2 (w_{pp} H_p + 2 w_p H_{pp} + w H_{ppp}) H =$$

$$= \kappa H G_1 - (\kappa U - \omega) G_2. \qquad (7.19)$$

From elimination of U_p from eqs. 7.7 and 7.8 follows

$$-(\kappa U-\omega)^2 H_p + \kappa^2 H H_p + \mu\kappa^2\omega^2 H H_{ppp} = 0,$$

and hence the terms with w in eq. 7.19 vanish. After some manipulations eq. 7.19 is written as

$$\mu\kappa\omega^2 \frac{\partial}{\partial p}\left[w_p H_p^2\right] = H_p G_1 + U_p G_2 = $$
$$= \mu\kappa\omega^2 \frac{\partial}{\partial p}\left[H_p H_{1p} - H_{pp} H_1\right].$$

Again the boundedness of the terms between square brackets requires that the integral over one period of $H_p G_1 + U_p G_2$ vanishes:

$$\int_0^1 (H_p G_1 + U_p G_2)dp = 0. \tag{7.20}$$

Partial integration gives:

$$\int_0^1 H_p G_1 dp = \int_0^1 H_p \left\{\int_p^p F_1(p')dp' + \gamma_1\right\} dp =$$
$$= \left[H\left\{\int_1^p F_1(p')dp' + \gamma_1\right\}\right]_0^1 - \int_0^1 HF_1 dp =$$
$$= -\int_0^1 HF_1 dp,$$

and similarly we have

$$\int_0^1 U_p G_2 dp = -\int_0^1 UF_2 dp.$$

Hence eq. 7.20 becomes

$$\int_0^1 (HF_1 + UF_2)dp = 0. \tag{7.21}$$

The 3 relations 7.17, 7.18 and 7.21 together with the dispersion relation 7.14 and the equation $\omega_x + \kappa_t = 0$ provide a set of five equations for the five unknown slowly varying functions $\omega(x,t)$, $\kappa(x,t)$, $\alpha(x,t)$, $\beta(x,t)$ and $\gamma(x,t)$

In close analogy to section 4 we can show that the integral relations 7.17, 7.18 and 7.21 can be obtained by an averaging of conservation laws. The Boussinesq equations 7.5 and 7.6 are conservation laws:

$$\frac{\partial}{\partial t}(u) + \frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2 + h + \frac{\mu}{K^2}h_{tt}\right) = 0, \tag{7.22}$$

$$\frac{\partial}{\partial t}(h) + \frac{\partial}{\partial x}(uh) = 0. \tag{7.23}$$

Equation 7.22 expresses the conservation of momentum and eq. 7.23 the conservation of mass. A third conservation law follows after multiplication of eq. 7.5 by h and eq. 7.6 by u:

$$h\left(u_t + uu_x + h_x + \frac{\mu}{K^2}h_{xtt}\right) + u\left(h_t + uh_x + hu_x\right) = 0, \tag{7.24}$$

or in conservative form:

$$\frac{\partial}{\partial t}\left(uh - \frac{\mu}{K^2}h_x h_t\right) + \frac{\partial}{\partial x}\left(hu^2 + \tfrac{1}{2}h^2 + \frac{\mu}{K^2}hh_{tt} + \frac{\mu}{2K^2}h_t^2\right) = 0. \tag{7.25}$$

From eq. 7.24 the structure of integral relation 7.21 becomes clear. If again a slowly varying wavetrain is considered locally as a uniform wavetrain with slowly varying parameters $\kappa(x,t)$, $\omega(x,t)$, $\alpha(x,t)$, $\beta(x,t)$ and $\gamma(x,t)$ we can put:

$$u = U(p,x,t) = U\left[p;\ \kappa(x,t),\ \omega(x,t),\ \alpha(x,t),\ \beta(x,t),\ \gamma(x,t)\right],$$

$$h = H(p,x,t) = H\left[p;\ \kappa(x,t),\ \omega(x,t),\ \alpha(x,t),\ \beta(x,t),\ \gamma(x,t)\right],$$

with

$$p_x = K\,\kappa(x,t) \text{ and } p_t = -\,K\,\omega(x,t).$$

Substitution into the conservation equations 7.22, 7.23 and 7.24 then yields equations of which the highest order terms (of order K) vanish by virtue of equations 7.7 and 7.8 that determine the periodic dependence of U and H on p. If the remaining parts are integrated over one period in p we successively arrive at:

$$\int_0^1 \left[U_t + U U_x + H_x + 2\mu\omega\omega_x H_{pp} - \mu\kappa\,\omega_t H_{pp} - 2\mu\kappa\,\omega H_{ppt} + \mu\omega^2 H_{ppx}\right] dp +$$

$$+\ O\left(\frac{1}{K}\right) = -\int_0^1 F_1\,dp + O\left(\frac{1}{K}\right) = 0,$$

$$\int_0^1 \left[H_t + UH_x + HU_x\right] dp = -\int_0^1 F_2\,dp = 0,$$

$$\int_0^1 \left[HF_1 + UF_2\right] dp + O\left(\frac{1}{K}\right) = 0.$$

These integral relations coincide, apart from terms order K^{-1} with eqs. 7.17, 7.18 and 7.21 respectively.

8. *Asymptotic expansion with respect to* ϵ.

The dependence of the wavetrain on the phase coordinate $p = KS(x,t)$ is governed by eqs. 7.12 and 7.13. As in the case of the cnoidal waves of Korteweg and de Vries the parameters α, β and γ have no clear physical meaning and in order to be able to express the equations in terms of amplitude, mean height, mean velocity etc. we put

$$H(p,x,t) = \bar{H}_0(x,t) + \bar{A}(x,t)\ \Phi\ (p,x,t), \tag{8.1}$$

$$U(p,x,t) = \bar{U}_0(x,t) + \bar{B}(x,t)\ \Psi\ (p,x,t), \tag{8.2}$$

with

$$\bar{H}_0(x,t) = \tfrac{1}{2}\left\{H_{max}(x,t) + H_{min}(x,t)\right\},$$

$$\bar{U}_0(x,t) = \tfrac{1}{2}\left\{U_{max}(x,t) + U_{min}(x,t)\right\}.$$

Then $\bar{A}$ and $\bar{B}$ can be considered as amplitude functions for the waveheight and velocity respectively. Φ and Ψ are periodic functions of p and the introduction of the parameters $\bar{H}_0$ and $\bar{U}_0$ allows us to take Φ and Ψ as functions oscillating between +1 and -1. Note, as in the case of the cnoidal waves, that $\bar{H}_0$ and $\bar{U}_0$ are not exactly equal to the mean elevation and mean velocity respectively. Substitution of eqs. 8.1 and 8.2 into eqs. 7.12 and 7.13 gives:

$$(-\omega + \kappa\bar{U}_o + \kappa\bar{B}\Psi)(\bar{H}_o + \bar{A}\Phi) = \beta, \tag{8.3}$$

$$\mu\kappa\omega^2\bar{A}^2\Phi_p^2 = \alpha(\bar{H}_o+\bar{A}\Phi) + \gamma + \frac{\beta^2}{\kappa(\bar{H}_o+\bar{A}\Phi)} - \kappa(\bar{H}_o+\bar{A}\Phi)^2 \tag{8.4}$$

From eq. 7.8 follows that H and U are oscillating in phase and hence Φ and Ψ reach their extremal values -1 and +1 simultaneously. Using this fact eqs. 8.3 and 8.4 give rise to four algebraical equations

$$\beta = (\bar{H}_o+\bar{A})\left[-\omega + \kappa(\bar{U}_o+\bar{B})\right] = (\bar{H}_o-\bar{A})\left[-\omega + \kappa(\bar{U}_o-\bar{B})\right],$$

$$\alpha(\bar{H}_o+\bar{A}) + \gamma + \frac{\beta^2}{\kappa(\bar{H}_o+\bar{A})} - \kappa(\bar{H}_o+\bar{A})^2 = 0,$$

$$\alpha(\bar{H}_o-\bar{A}) + \gamma + \frac{\beta^2}{\kappa(\bar{H}_o-\bar{A})} - \kappa(\bar{H}_o-\bar{A})^2 = 0.$$

From the first of these equations follows

$$\kappa\bar{H}_o\bar{B} - \bar{A}\omega + \bar{A}\kappa\bar{U}_o = 0, \tag{8.5}$$

which is one of the two dispersion relations for the problem. Solving α, β and γ as functions of $\bar{H}_o$, $\bar{U}_o$, $\bar{A}$ and $\bar{B}$ and substituting them into eq. 8.4 we find

$$\mu\omega^2\Phi_p^2 = (1-\Phi^2)\left[1 - \frac{\bar{B}^2(\bar{H}_o^2-\bar{A}^2)}{\bar{A}^2(\bar{H}_o+\bar{A}\Phi)}\right]. \tag{8.6}$$

The order of magnitude of the parameters $\bar{A}, \bar{B}, \bar{H}_o$ and $\bar{U}_o$ in eq. 8.6 is expressed by the substitutions:

$$\bar{A} = \epsilon A, \quad \bar{B} = \epsilon B, \quad \bar{H}_o = 1+\epsilon^2\eta_1, \quad \bar{U}_o = \epsilon^2 u_1. \tag{8.7}$$

with A, B, η_1 and u_1 quantities of order unity.
Equation 8.5 then yields.

$$\frac{B}{A} = \frac{\omega}{\kappa} + O(\epsilon^2). \tag{8.8}$$

After substitution of eqs. 8.7 into eq. 8.6 and expanding straightforwardly with respect to ϵ and using eq. 8.8 in order to eliminate B, we get the following differential equation for $\Phi(p,x,t)$:

$$\mu\kappa^2\omega^2\Phi_p^2 = (1-\Phi^2)\Big[\kappa^2-\omega^2+\epsilon A\omega^2\Phi + $$
$$+ \epsilon^2(2\kappa\omega u_1 + A^2\omega^2 - A^2\omega^2\Phi^2 + \omega^2\eta_1) + O(\epsilon^3)\Big]. \tag{8.9}$$

This equation is the counterpart of eq. 5.4 for the Korteweg- de Vries waves. The treatment of this equation with Lindstedt's method in order to obtain an asymptotic expansion with respect to ϵ of $\Phi(p,x,t)$ in the form of a Fourier series is completely analogous to section 5 and will not be repeated here. The resulting expansion of H(p, x, t) becomes:

$$H = 1 + \epsilon A\cos(2\pi p) + \epsilon^2\eta_1 + \frac{\epsilon^2A^2}{16\pi^2\mu\kappa^2}\left[\cos(4\pi p) - 1\right] + O(\epsilon^3). \tag{8.10}$$

The condition of periodicity in p with period 1 yields the dispersion relation

$$\kappa^2 - \omega^2 = 4\pi^2 \mu \kappa^2 \omega^2 + \epsilon^2 \left[-2\omega\kappa u_1 - \tfrac{1}{2} A^2 \omega^2 - \omega^2 \eta_1 + \frac{3A^2\omega^2}{32\pi^2 \mu \kappa^2} \right] + O(\epsilon^3). \quad (8.11)$$

The corresponding series expansion of U(p, x, t) is obtained by using eq. 7.12 and expanding with respect to ϵ:

$$U = \epsilon \frac{A\omega}{\kappa} \cos(2\pi p) + \epsilon^2 \left[u_1 + \frac{A^2\omega}{2\kappa} - \frac{A^2\omega}{16\pi^2 \mu \kappa^3} + \left(\frac{A^2\omega}{16\pi \mu \kappa^3} - \frac{A^2\omega}{2\kappa} \right) \times \right.$$

$$\left. \times \cos(4\pi p) \right] + O(\epsilon^3). \quad (8.12)$$

Introducing the mean elevation H_o and the mean velocity U_o by means of

$$H_o = \epsilon^2 \left[\eta_1 - \frac{A^2}{16\pi^2 \mu \kappa^2} \right],$$

$$U_o = \epsilon^2 \left[u_1 + \frac{A^2\omega}{2\kappa} - \frac{A^2\omega}{16\pi^2 \mu \kappa^3} \right],$$

the expressions for H and U and the dispersion relation 8.11 become finally:

$$H = 1 + H_o + \epsilon A \cos(2\pi p) + \frac{\epsilon^2 A^2}{16\pi^2 \mu \kappa^2} \cos(4\pi p) + O(\epsilon^3), \quad (8.13)$$

$$U = U_o + \epsilon \frac{A\omega}{\kappa} \cos(2\pi p) + \epsilon^2 \left[\frac{A^2\omega}{16\pi^2 \mu \kappa^3} - \frac{A^2\omega}{2\kappa} \right] \cos(4\pi p) + O(\epsilon^3), \quad (8.14)$$

$$\kappa^2 - \omega^2 = 4\pi^2 \mu \kappa^2 \omega^2 - 2\omega\kappa U_o - \omega^2 H_o + \epsilon^2 \left[\tfrac{1}{2} A^2 \omega^2 - \frac{3A^2\omega^2}{32\pi^2 \mu \kappa^2} \right] + O(\epsilon^3). \quad (8.15)$$

Dispersion relation 8.15 is written more conveniently as

$$\omega = \omega_o(\kappa) + \frac{H_o \omega_o^3}{2\kappa^2} + \frac{U_o \omega_o^2}{\kappa} - \frac{\omega_o^3 E}{4\kappa^2} \left[1 - \frac{3}{16\pi^2 \mu \kappa^2} \right], \quad (8.16)$$

with $\omega_o(\kappa) = \kappa / \sqrt{1 + 4\pi^2 \mu \kappa^2}$ and $E = \epsilon^2 A^2$

The series expansions 8.13 and 8.14 for H(p, x, t) and U(p, x, t) are substituted into the integral relations 7.17, 7.18 and 7.21. Integral relation 7.17 is written as

$$\frac{\partial}{\partial t} \left[\int_0^1 U dp \right] + \frac{\partial}{\partial x} \left[\tfrac{1}{2} \int_0^1 U^2 dp \right] + \frac{\partial}{\partial x} \left[\int_0^1 H dp \right] +$$

$$+ 2\omega\omega_x \mu \left[\int_0^1 H_{pp} dp \right] - \mu\kappa\omega_t \left[\int_0^1 H_{pp} dp \right] - 2\kappa\omega\mu \left[\int_0^1 H_{ppt} dp \right] +$$

$$+ \omega^2 \mu \left[\int_0^1 H_{ppx} dp \right] = 0.$$

Because of the periodicity in p of H_p, H_{px} and H_{pt} the last four terms vanish and substitution of 8.13 and 8.14 gives

$$\frac{\partial U_o}{\partial t} + \frac{\partial}{\partial x}\left[H_o + \frac{E\,\omega_o^2}{4\,\kappa^2}\right] + O(\epsilon^4) = 0. \qquad (8.17)$$

Integral relation 7.18 is recasted as

$$\frac{\partial}{\partial t}\left[\int_0^1 H dp\right] + \frac{\partial}{\partial x}\left[\int_0^1 U\,H dp\right] = 0$$

and gives after substitution of 8.13 and 8.14.

$$\frac{\partial H_o}{\partial t} + \frac{\partial}{\partial x}\left[U_o + \frac{E\,\omega_o}{2\kappa}\right] + O(\epsilon^4) = 0. \qquad (8.18)$$

Integral relation 7.21 reads:

$$\frac{\partial}{\partial t}\left[\int_0^1 U\,H dp\right] + \frac{\partial}{\partial x}\left[\int_0^1 U^2 H dp\right] + \frac{\partial}{\partial x}\left[\tfrac{1}{2}\int_0^1 H^2 dp\right] +$$

$$+ \mu(2\,\omega\,\omega_x - \kappa\omega_t)\left[\int_0^1 H\,H_{pp} dp\right] - 2\mu\kappa\omega\left[\int_0^1 H\,H_{ppt}\,dp\right] +$$

$$+ \mu\omega^2\left[\int_0^1 H\,H_{ppx}\,dp\right] = 0,$$

and yields after substitution of U and H use of eq. 8.17:

$$\frac{\partial}{\partial t}\left\{\frac{A^2\omega}{2\kappa}\right\} + \frac{\partial}{\partial x}\left\{\frac{A^2\omega^2}{4\kappa^2} + \tfrac{1}{4}A^2\right\} + 2\pi^2\mu\left[(\kappa\,\omega_t - 2\,\omega\,\omega_x)A^2 + \right.$$

$$\left. + 2\kappa\omega A A_t - \omega^2 A A_x\right] + O(\epsilon^2) = 0. \qquad (8.19)$$

The term between square brackets is equal to

$$\frac{\partial}{\partial t}(\kappa\omega A^2) - \frac{\partial}{\partial x}(\tfrac{1}{2}A^2\omega^2),$$

and using the dispersion relation eq. 8.19 can be transformed into

$$\frac{\partial}{\partial t}\left[\frac{A^2\kappa}{\omega_o}\right] + \frac{\partial}{\partial x}\left[\frac{A^2\,\omega_o^2}{\kappa^2}\right] + O(\epsilon^2) = 0.$$

Using $\omega_o'(\kappa) = \omega_o^3/\kappa^3$, we arrive at

$$\frac{\partial}{\partial x}\left[\frac{E\,\kappa}{\omega_o}\right] + \frac{\partial}{\partial x}\left[\omega_o'(\kappa)\,\frac{E\,\kappa}{\omega_o}\right] + O(\epsilon^4) = 0. \qquad (8.20)$$

By virtue of

$$\frac{\partial}{\partial t}\left(\frac{\kappa}{\omega_o}\right) + \omega_o'(\kappa)\,\frac{\partial}{\partial x}\left(\frac{\kappa}{\omega_o}\right) = \left[\frac{1}{\omega_o} - \frac{\kappa\,\omega_o'(\kappa)}{\omega_o^2}\right]\left[\kappa_t + \omega_o'(\kappa)\kappa_x\right] =$$

$$= \left[\frac{1}{\omega_o} - \frac{\kappa\,\omega_o'(\kappa)}{\omega_o^2}\right]\frac{\partial}{\partial x}\left[\omega - \omega_o(\kappa)\right] = O(\epsilon^2),$$

equation 8.20 is equivalent to the same order of approximation with:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\omega_o'(\kappa)E\right] = 0. \qquad (8.21)$$

This equation which expresses the conservation of "averaged energy" E propagating with velocity $C_o = \omega_o'(\kappa)$ is of the same form as the "energy equation" 5.13 for the Korteweg- de Vries waves.

We have obtained a set of five equations for the five slowly varying functions $\kappa(x,t)$, $\omega(x,t)$, $E(x,t)$, $U_o(x,t)$, and $H_o(x,t)$, viz. eqs. 8.17, 8.18, 8.21, the dispersion relation 8.16 and the relation $\omega_x + \kappa_t = 0$. In the next section we will investigate these equations further, and it will appear that they can be reduced to a similar system as eqs. 6.9 and 6.10 for the cnoidal waves.

9. *Reduction to a system of equations for* κ *and* E *only.*

The calculation of the four characteristic velocities C_i of eqs. 8.16, 8.17, 8.19 and 8.21 proceeds along similar lines as in section 6. Writing eq. 8.16 as

$$\kappa_t + C_o\kappa_x + \alpha_1 E_x + \alpha_1' E\kappa_x + \alpha_2 H_{ox} + \alpha_2' H_o\kappa_x + \alpha_3 U_{ox} + \alpha_3' U_o\kappa_x = 0, \qquad (9.1)$$

where the prime indicates differentiation with respect to κ and where

$$\alpha_1 = \frac{\omega_o^3}{4\kappa^2}\left(\frac{3}{16\pi^2\mu\kappa^2} - 1\right),$$

$$\alpha_2 = \frac{\omega_o^3}{2\kappa^2}, \quad \alpha_3 = \frac{\omega_o^2}{\kappa},$$

we add eqs. 8.17, 8.18 and 8.21, multiplied with factors λ, σ and ν respectively, to eq. 9.1. The condition that κ, E, U_o and H_o are differentiated in the same characteristic direction yields four algebraical equations for C, λ, σ and ν:

$$C = C_o + \nu E C_o' + \alpha_1' E + \alpha_2' H_o + \alpha_3' U_o + \lambda\beta' E + \sigma\gamma' E,$$

$$\nu C = \alpha_1 + \lambda\beta + \sigma\gamma + \nu C_o,$$

$$\lambda C = \sigma + \alpha_3,$$

$$\sigma C = \lambda + \alpha_2,$$

with abbreviations:

$$\beta = \frac{\omega_o^2}{4\kappa^2}, \quad \gamma = \frac{\omega_o}{2\kappa}.$$

Elimination of λ, σ and ν yields one equation for C:

$$(C-C_o)^2 = (C-C_o)\left[\alpha_1' E + \alpha_2' H_o + \alpha_3' U_o + \frac{E}{C^2-1}\left\{\beta'(\alpha_2+\alpha_3 C) + \gamma'(\alpha_3+\alpha_2 C)\right\}\right] + E C_o'\left[\alpha_1 + \frac{1}{C^2-1}\left\{\beta(\alpha_2+\alpha_3 C) + \gamma(\alpha_3+\alpha_2 C)\right\}\right]. \qquad (9.2)$$

For linear waves $E\to 0$, $H_o\to 0$, $U_o\to 0$, and then we have again a double root $C = C_o = \omega_o'(\kappa)$, corresponding to the linear group velocity. For non-linear waves with ϵ small, it is seen from eq. 9.2 that there are two roots C_1 and C_2 near C_o and two other roots C_3 and C_4 near -1 and +1 respectively. Approximately we have

$$C_{1,2} = C_o \pm \sqrt{E\,C_o'\left[\alpha_1 + (C_o^2-1)^{-1}\left\{\beta(\alpha_2+\alpha_3 C_o) + \gamma(\alpha_3+\alpha_2 C_o)\right\}\right]} + O(\epsilon^{\frac{3}{2}})$$

$$= C_o \pm \sqrt{\frac{E\,C_o'\,\omega_o^9}{16\,\kappa^8(C_o^2-1)}\left[12\left(\frac{\kappa}{\omega_o}\right)^6 - \left(\frac{\kappa}{\omega_o}\right)^4 + 5\left(\frac{\kappa}{\omega_o}\right)^2 - 7\right]} + O(\epsilon^{\frac{3}{2}}). \qquad (9.3)$$

By virtue of $\omega_o = \kappa/\sqrt{1+4\pi^2\mu\kappa^2}$, the following inequalities hold:

$$C_o'(\kappa) = \omega_o''(\kappa) < 0,\ C_o^2 - 1 = \left[\omega_o'(\kappa)\right]^2 - 1 < 0,\ \frac{\kappa}{\omega_o} > 1.$$

Furthermore the polynomial of the sixth degree in κ/ω_o between square brackets in eq. 9.3 is positive for $\kappa/\omega_o > 1$ and hence the square root is real. So there are always two distinct real roots near C_o. The two remaining roots C_3 and C_4 will not be given here; it is sufficient to know that they are real and of order of magnitude $\pm 1 + O(\epsilon^2)$. So we have a purely hyperbolic system, which is written as

$$\left[\kappa_t + C_i\kappa_x\right] + \nu_i\left[E_t + C_i E_x\right] + \sigma_i\left[H_{ot} + C_i H_{ox}\right] + \lambda_i\left[U_{ot} + C_i U_{ox}\right] = 0,$$
$$(i = 1, 2, 3, 4) \qquad (9.4)$$

with multipliers ν_i, σ_i and λ_i given by

$$\nu_i = \frac{1}{C_i - C_o}\left[\alpha_1 + \frac{1}{C_o^2 - 1}\left\{\beta(\alpha_2+\alpha_3 C_i) + \gamma(\alpha_3+\alpha_2 C_i)\right\}\right],$$

$$\sigma_i = \frac{\alpha_3 + \alpha_2 C_i}{C_i^2 - 1},$$

$$\lambda_i = \frac{\alpha_2 + \alpha_3 C_i}{C_i^2 - 1}.$$

As in the case of the cnoidal waves of Korteweg and de Vries we consider only i=1 and i=2, viz. the characteristic velocities lying near the group velocity C_o. For i=1 and i=2 we observe that ν_i is large of order ϵ^{-1} compared to λ_i and σ_i and hence the terms with U_o and H_o can be omitted from eq. 9.4 for i=1 and i=2. Noting also that C_1 and C_2 do not depend on H_o and U_o to the present order of approximation, we get for i=1 and i=2 a set of two equations for $\kappa(x,t)$ and $E(x,t)$ only:

$$\kappa_t + \left\{C_o + \sqrt{E\,C_o'F(\kappa)}\right\}\kappa_x + \frac{F(\kappa)}{\sqrt{C_o'\,E\,F(\kappa)}}\left[E_t + \left\{C_o + \sqrt{E\,C_o'F(\kappa)}\right\}E_x\right] = 0, \quad (9.5)$$

$$\kappa_t + \left\{C_o - \sqrt{E\,C_o'F(\kappa)}\right\}\kappa_x - \frac{F(\kappa)}{\sqrt{C_o'\,E\,F(\kappa)}}\left[E_t + \left\{C_o - \sqrt{E\,C_o'F(\kappa)}\right\}E_x\right] = 0, \quad (9.6)$$

where we have used the abbreviation:

$$F(\kappa) = \frac{\omega_o^9(\kappa)}{16\,\kappa^8\left[C_o^2(\kappa) - 1\right]}\left[12\left(\frac{\kappa}{\omega_o}\right)^6 - \left(\frac{\kappa}{\omega_o}\right)^4 + 5\left(\frac{\kappa}{\omega_o}\right)^2 - 7\right] < 0.$$

Addition and subtraction of eqs. 9.5 and 9.6 yields the set of equations

$$\frac{\partial \kappa}{\partial t} + C_o(\kappa)\frac{\partial \kappa}{\partial x} + F(\kappa)\frac{\partial E}{\partial x} = 0, \tag{9.7}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[C_o(\kappa)E\right] = 0. \tag{9.8}$$

Introducing C_o as a new dependent variable eq. 9.7 becomes

$$\frac{\partial C_o}{\partial t} + C_o\frac{\partial C_o}{\partial x} + f(C_o)\frac{\partial E}{\partial x} = 0, \tag{9.9}$$

with $f(C_o)$ determined by

$$f\left[C_o(\kappa)\right] = C_o'(\kappa)\, F(\kappa).$$

To the same order of approximation eq. 9.9 can be replaced by

$$\frac{\partial C_o}{\partial t} + C_o\frac{\partial C_o}{\partial x} + \frac{\partial}{\partial x}\left\{f(C_o)E\right\} = 0, \tag{9.10}$$

because the additional term $f'(C_o)E\,C_{ox}$ in eq. 9.10. only contributes to terms of order $\epsilon^{3/2}$ in the characteristic velocities and hence to the present order of approximation the set of equations 9.10 and 9.8 have the same characteristic form as eqs. 9.9 and 9.8:

$$C_{ot} + \left\{C_o \pm \sqrt{Ef(C_o)}\right\}C_{ox} \pm \sqrt{f(C_o)/E}\left[E_t + \left\{C_o \pm \sqrt{Ef(C_o)}\right\}E_x\right] = 0.$$

Putting now $\bar{E} = Ef(C_o)$, we have the set of equations

$$\frac{\partial C_o}{\partial t} + C_o\frac{\partial C_o}{\partial x} + \frac{\partial \bar{E}}{\partial x} = 0, \tag{9.11}$$

$$\frac{\partial}{\partial t}\left[\frac{\bar{E}}{f(C_o)}\right] + \frac{\partial}{\partial x}\left[C_o\frac{\bar{E}}{f(C_o)}\right] = 0,$$

or

$$\frac{\partial \bar{E}}{\partial t} + \frac{\partial}{\partial x}\left[C_o\bar{E}\right] - \frac{\bar{E}\,C_o'(\kappa)}{f(C_o)}\left[\kappa_t + C_o\kappa_x\right] =$$

$$= \frac{\partial \bar{E}}{\partial t} + \frac{\partial}{\partial x}(C_o\bar{E}) + O(\epsilon^4) = 0, \tag{9.12}$$

which is similar to eqs. 6.9 and 6.10 for the cnoidal waves. Hence also for Boussinesq waves we have the gas dynamics analogy for the "energy density" function $\bar{E}(x,t)$ and its propagation velocity $C_o(x,t)$

10. *Explicit solution of the dispersion equations*.

The dispersion equations 6.9 and 6.10 for the cnoidal waves and 9.11 and 9.12 for the Boussinesq waves both have the same form as the equations for the unsteady one-dimensional motion of a compressible gas with a fictitious adiabatic relation between pressure and density. In order to stress this analogy we write the equations as

$$u_t + u\,u_x + \epsilon^2\rho_x = 0, \tag{10.1}$$

$$\rho_t + (\rho u)_x = 0, \tag{10.2}$$

with $\rho(x,t)$ and $u(x,t)$ of order of magnitude one. Introduction of a new variable $v(x,t)$ by

$$v(x,t) = 2\epsilon\sqrt{\rho(x,t)},$$

yields the equations

$$u_t + u u_x + \tfrac{1}{2} v v_x = 0, \qquad (10.3)^a$$

$$v_t + \tfrac{1}{2} v u_x + u v_x = 0. \qquad (10.3)^b$$

If we consider x and t as functions of u and v, eqs. 10.3^a and 10.3^b are inverted into

$$-x_v + u t_v - \tfrac{1}{2} v t_u = 0, \qquad (10.4)$$

$$x_u + \tfrac{1}{2} v t_v - u t_u = 0. \qquad (10.5)$$

Introducing a function $\Phi = \Phi(u,v)$ by

$$\frac{\partial\Phi}{\partial v} = -\tfrac{1}{2} v t, \quad \frac{\partial\Phi}{\partial u} = x - ut, \qquad (10.6)$$

it is seen that eq. 10.4 is satisfied. In order to satisfy eq. 10.5. the function $\Phi(u,v)$ must satisfy the axisymmetric wave equation

$$\Phi_{vv} + \frac{1}{v}\Phi_v - \Phi_{uu} = 0. \qquad (10.7)$$

Let us consider the case of an initially given slowly varying wavetrain, i.e. $u(x,o)$ and $\rho(x,o)$ are given functions of x. Elimination of x between $u(x,o)$ and $\rho(x,o)$ gives a curve in the u,v-plane that has a distance of order ϵ from the u-axis, because in our analysis we always have $v=O(\epsilon)$. On this curve, $v=\epsilon g(u)$ say, we have $t=0$ and x is a given function of u, say $x=x_o(u)$. From eqs. 10.6 follows that on this curve $v=\epsilon g(u)$ the function $\Phi(u,v)$ must satisfy

$$\Phi_v = 0, \qquad \Phi_u = x_o(u).$$

In this way a boundary value problem for eq. 10.7 is formulated which bears some resemblance to the axisymmetric slender body theory in supersonic aerodynamics. As we are only interested in solutions for small values (of order ϵ) of v, this problem can be solved approximately in a simple way by replacing the term Φ_{uu} in eq. 10.7 by $x_o'(u)$ for small values of v. Then eq. 10.7 reduces to an ordinary differential equation which is solved easily:

$$\Phi(u,v) \approx X_o(u) + \tfrac{1}{2}\epsilon^2 g^2(u)\, X_o''(u) \ln\left[\frac{\epsilon g(u)}{v}\right] + \tfrac{1}{4} X_o''(u)\left[v^2 - \epsilon^2 g^2(u)\right],$$

where

$$X_o'(u) = x_o(u).$$

Lighthill [13], proceeding from the averaged Lagrangian principle of Whitham, also arrived at the axisymmetric wave equation 10.7 in the case of absence of so-called pseudo-frequencies, such as mean waveheight, mean velocity etc. In fact this paper shows that for cnoidal waves and Boussinesq waves a similar theory as expounded by Lighthill is possible and hence for a detailed study of boundary value problems arising from eq. 10.7 in the case of an initially given slowly varying wavetrain and also in the case of various kinds of wave-makers, we refer to Lighthill [13].

References.

1. Eckart, C — The approximate solution of one-dimensional wave equations. Rev. Mod. Phys., Vol. 20, no. 2, pp. 399-417, 1948
2. Lighthill, M.J. — Group Velocity, J. Inst. Maths. Appl., Vol I, pp. 1-28, 1965
3. Jeffreys, H and Jeffreys, B.S. — Methods of mathematical physics, Cambridge Univ. Press, 1964
4. Peletier, L.A. — On a class of wave equations, Thesis Eindhoven, 1967
5. Brillouin, L and Sommerfeld, A. — Wave propagation and group velocity, Academic Press, New York, 1960.
6. Lewis, R.M. — Asymptotic methods for the solution of dispersive hyperbolic equations. In: "Asymptotic Solutions of differential equations and their applications", C.H. Wilcox, e.d., University of Wisconsin Symposium Proceedings, pp. 53-107, Wiley, New York, 1964
7. Bleistein, N and Lewis, R.M. — Space-time diffraction for dispersive hyperbolic equations. SIAM. J. Appl. Math. Vol. 14, No. 6, pp. 1454-1470, 1966
8. Boersma, J. — Uniform asymptotic solutions of discontinuous initial value problems for dispersive hyperbolic equations. Quart. Appl. Math., Vol. 25, No. 1, pp. 31-44, 1967
9. Whitham, G.B. — Non-linear dispersive waves. Proc. Roy. Soc. A, 283, pp. 238-261, 1965
10. Whitham, G.B. — A general approach to linear and non-linear dispersive waves using a Lagrangian. J. Fluid Mech., Vol. 22, 273-283, 1965
11. Whitham, G.B. — Non-linear dispersion of water waves. J. Fluid Mech., Vol. 27, pp. 399-412, 1967
12. Whitham, G.B. — Variational methods and applications to water waves. Proc. Roy. Soc. A, 299, pp. 6-25, 1967
13. Lighthill, M.J. — Contributions to the theory of waves in non-linear dispersive systems. J. Inst. Math. Appl., Vol 1, pp. 269-306, 1965
14. Lighthill, M.J. — Some special cases treated by the Whitham theory. Proc. Roy. Soc., A, 299, pp. 28-53, 1967
15. Luke, J.C. — A perturbation method for non-linear dispersive wave problems. Proc. Roy. Soc., A, 292, pp. 403-412, 1966.
16. Korteweg, D.J. and de Vries, G. — On the change of form of long waves advancing in a rectangular canal. Phil. Mag. (5), vol. 39, pp. 422-443, 1895
17. Minorsky, N. — Non-linear oscillations, van Nostrand, Princeton, 1962.

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